Annals of Mathematics and Artificial Intelligence **34:** 131–159, 2002.

A qualitative approach to syllogistic reasoning

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In this paper we present a new approach to a symbolic treatment of quantified statements having the following form "Q A's are B's", knowing that A and B are labels denoting sets, and Q is a linguistic quantifier interpreted as a proportion evaluated in a qualitative way. Our model can be viewed as a symbolic generalization of statistical conditional probability notions as well as a symbolic generalization of the classical probabilistic operators. Our approach is founded on a symbolic finite M-valued logic in which the graduation scale of M symbolic quantifiers is translated in terms of truth degrees. Moreover, we propose symbolic inference rules allowing us to manage quantified statements.

Keywords: knowledge representation, statistical information, linguistic quantifiers, quantified assertions, syllogistic reasoning

1. Introduction

In the natural language, one often uses statements qualifying statistical information like "Most students are single", called *quantified assertions*. More formally, these quantified assertions have the following form "Q A's are B's", where A and B are labels denoting sets, and Q is a *linguistic quantifier* interpreted as a proportion evaluated in a qualitative way. In this paper, we propose a new approach to such quantified assertions within a *qualitative context.* More precisely, our goal is twofold: (1) to propose a *symbolic representation* of *quantified assertions* and (2) to developp *syllogistic reasoning* allowing us to imply new quantified assertions from the initial ones.

Zadeh [35] distinguishes two types of *quantifiers: absolute* and *proportional*. An absolute quantifier evaluates the number of individuals of B in A. While a proportional quantifier evaluates the proportion of individuals of B in A. The proportional quantifiers can be precise or vague. A *precise quantifier* translates an interval of proportions having precise bounds exemplified by "10%", "Between 10 and 20%", etc. While a *vague quantifier* translates an interval proportions having fuzzy bounds. Thus the vague proportional quantifiers express qualitatively proportions. A *proportional quantifier* can be viewed as a kind of probabilities assigned to classes of individuals. So, several approaches based on the theory of probabilities have been proposed [3,4,6,11,20,29] for the modeling of precise proportional quantifiers. Other probabilistic approaches, such as those proposed by [5,23,24], do not enable an adequate representation of proportional quantifiers, since these approaches are generally introduced to treat uncertainty. These authors interpret the probability degrees assigned to propositions as degrees of certainty or beliefs in the truth of these propositions. They represent statistical assertions of type "Q A's are B's" as uncertain rules of the form: "if A then B" with a belief degree in the truth of the rule (A and B are interpreted as propositions). It has been pointed out by Bacchus [3] that a confusion in the representation is made between the probabilities interpreted as certainty degrees assigned to propositions about particular individuals and those interpreted as proportions assigned to classes of individuals. The *probabilities* of the first type are called *subjective* and the second *statistical.* The statistical probability that corresponds to the proportion is a particular case of probabilities where the distribution is uniform over the finite reference set. For example, the statistical probability attached to a subset A of the finite reference set Ω, $\mathrm{Prop}(A)$, is equal to the absolute proportion of individuals of A, i.e., $\mathrm{Prop}(A) = |A|/|\Omega|$. Similarly, if A and B are two subsets of Ω, the relative proportion of individuals of B in A is expressed by the conditional statistical probability, $\mathrm{Prop}(B|A)$, with $\mathrm{Prop}(B|A) = \mathrm{Prop}(A \cap B)/\mathrm{Prop}(A) = |A \cap B|/|A|$. Some probabilistic approaches [1,3,4,28] are interesting in a qualitative modeling of the proportional quantifier "Most" or "Almost-all" in the context of default reasoning. The approaches based on the fuzzy set theory [6,9,10,32,33,35] deal with a vague proportional quantifier as a fuzzy number of the interval [0, 1] which can be manipulated by using the fuzzy arithmetic. For example, the membership function of "Most" evaluates the degree to which a given proportion r is compatible with the quantifier "Most". The representation of quantified statements involving fuzzy sets is based on the concept of fuzzy subset cardinality. Recently, Dubois et al. [7] have proposed a semi-numerical approach to vague quantifiers based upon the numerical results obtained in [6] for precise quantifiers. It is concerned with a suitable ordered partition of the unit interval [0, 1] in several subintervals covering [0, 1], each subinterval representing a vague quantifier. The subintervals obtained by applying the inference rules (on the precise quantifiers) to subintervals representing the vague quantifiers are approximately associated to subintervals of vague quantifiers.

Before giving the content of this paper, let us stand some precision upon the information we want to deal with here. The initial context of our approach is to study the representation and the management of knowledge based on imprecise, incomplete or uncertain information. Moreover, we wish results in good accordance with the common sense reasoning. In this case, information are often evaluated in a qualitative rather than a quantitative way. Indeed, the human understanding of graduations scales of vagueness, certainty, incompleteness is generally better when the scales of valuation are discrete and qualitative. More precisely, one is not able to understand the meaning of an infinite number of degrees, and one cannot recognize the slight difference between two numerical degrees like 0.74 and 0.75. On the other hand, one will understand qualitative valuations using few adverbial expressions like "very little", "most", "almost all", etc.

It is clear that a purely symbolic framework is interesting when the information is not quantitative. If we consider information expressed with linguistic quantifiers, we do not know exactly to what numbers they refer to but give imprecise information about them by the way of a (finite) scale of values. The solution when dealing with this kind of information is either to translate the linguistic data into numerical data or to keep linguistic data without the help of numerical properties. Unfortunately, it can be noted that it is impossible to construct a quantitative/qualitative interface leading, in all cases, to satisfactory results. That is why, *our system is basically devoted to the common sense reasoning referring to information evaluated in a qualitative way*, like it is the case when information is expressed in the naturel language. It can also be applied to some problems using data of the numerical interval [0, 1] for which a finite partition can be linguistically accepted. In such cases, each subinterval can be defined by a linguistic expression. Then, we propose to give up quantitative values and to rather develop a purely symbolic framework. The goal of this paper is then to provide a method to deal with qualitative information linked to common sense reasoning.

In the first part of this paper (devoted to the first goal of our work), we present a *symbolic representation of quantified assertions*, and their fundamental properties. In section 2, we briefly present the M-valued predicate Logic proposed by Pacholczyk in [25–27]. Section 3, describes our symbolic representation of statistical probability. The axioms governing this representation are presented in section 4. Section 5 deals with certain properties generalizing symbolically some classical properties. Section 6 is devoted to the second goal of our paper [13–18], that is to say, *syllogistic reasoning* based on quantified assertions. Finally, in section 7 we make link with probabilistic works of Bacchus [3].

2. Informal introduction to many-valued predicate calculus

The representation of quantified assertions use a logical framework that deals with information in a purely symbolic way. In this section, we give the intuitive ideas governing this logic which are necessary for the understanding of the rest of the article.

The main difference between a many-valued predicate logic and a classical predicate logic is that it has more than two truth degrees (in our logic, we will use seven degrees: {not at all-true, very little-true, little-true, moderately-true, very-true, almost-true, totally-true}, the first degree and the last one corresponding respectively to false and true).

Since there are several truth degrees, a formula is not necessary true or false but true with a particular degree. So, a notion of partial satisfaction has been defined. We will particularly use this definition to describe the notion of proportion. The partial satisfaction means that an interpretation verifies a formula *with respect to a truth degree*.

Let $\mathfrak{L}$ be the many-valued predicates language and $\mathfrak{F}$ the set of formulas of $\mathfrak{L}$. We call an *interpretation structure* $\mathfrak{A}$ of $\mathfrak{L}$, the pair $\langle \mathfrak{D}, \{R_n \mid n \in \mathbb{N}\}\rangle$, where $\mathfrak{D}$ designates the domain of $\mathfrak{A}$ and R_n designates the *multiset*[1] associated with the predicate P_n of the language. We call a *valuation* of variables of $\mathfrak{L}$, a sequence de-

[1] The multiset theory is an axiomatic approach to the fuzzy set theory. In this theory, $x \in_\alpha A$, the membership degree to which x belongs to A, corresponds with $\mu_A(x) = \alpha$ in the fuzzy set theory of Zadeh [34].

noted as $s = \langle s_0, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots\rangle$. The valuation $s(i/a)$ is defined by $s(i/a) = \langle s_0, \ldots, s_{i-1}, a, s_{i+1}, \ldots\rangle$.

Definition 1. For any formula Φ of F, the relation of partial satisfaction "s satisfies Φ to a degree τ_α in-$\mathfrak{A}$" or "s τ_α-satisfies Φ in-$\mathfrak{A}$", denoted as $\mathfrak{A} \models^s_\alpha \Phi$, is defined recursively as follows:

- $\mathfrak{A} \models^s_\alpha P_n(z_{i_1}, \ldots, z_{i_k}) \Leftrightarrow \langle s_{i_1}, \ldots, s_{i_k}\rangle \in_\alpha R_n$,
- $\mathfrak{A} \models^s_\alpha \neg\phi \Leftrightarrow \{\mathfrak{A} \models^s_\beta \phi$ with $\tau_\alpha = \tau_{M+1-\beta}\}$,
- $\mathfrak{A} \models^s_\alpha \phi \cap \psi \Leftrightarrow \{\mathfrak{A} \models^s_\beta \phi$ and $\mathfrak{A} \models^s_\gamma \psi$ with $\tau_\alpha = \tau_\beta \wedge \tau_\gamma\}$,
- $\mathfrak{A} \models^s_\alpha \phi \cup \psi \Leftrightarrow \{\mathfrak{A} \models^s_\beta \phi$ and $\mathfrak{A} \models^s_\gamma \psi$ with $\tau_\alpha = \tau_\beta \vee \tau_\gamma\}$,
- $\mathfrak{A} \models^s_\alpha \phi \supset \psi \Leftrightarrow \{\mathfrak{A} \models^s_\beta \phi$ and $\mathfrak{A} \models^s_\gamma \psi$ with $\tau_\alpha = \tau_\beta \rightarrow \tau_\gamma\}$,
- $\mathfrak{A} \models^s_\alpha \exists z_n\psi \Leftrightarrow \tau_\alpha = \mathrm{Max}\{\tau_\gamma \mid \mathfrak{A} \models^{s(n/a)}_\gamma \psi, a \in D\}$,
- $\mathfrak{A} \models^s_\alpha \forall z_n\psi \Leftrightarrow \tau_\alpha = \mathrm{Min}\{\tau_\gamma \mid \mathfrak{A} \models^{s(n/a)}_\gamma \psi, a \in D\}$.

In this logic, we will define a new predicate (called **Prop**) to describe the proportion (section 3). The truth degree of **Prop** will be interpreted as a symbolic proportion.

We can now explain why we refer to an M-valued predicate logic to deal with vagueness, incompleteness and uncertainty, rather than to use a modal logic for the management of these notions, like it is the case in some probabilistic logic [8]. Basically we use the fact that the M-valued predicate logic used here [25–27], has been constructed to represent imprecise properties. Then, we can keep working in this M-valued logic to represent the notion of statistical probabilities by adding a specific predicate and by defining the semantical axiomatics governing them. More precisely, the scale of basic truth degrees is associated with each of the proportions scale.

Moreover, we use the M-valued logic since it is a logical framework where the necessary properties of a logical framework (soundness, consistency, completeness) have been proved.

3. Symbolic representation of statistical probabilities

The representation of statistical probabilities requires the reference to sets of individuals and also to assign probabilities to these sets. Let us denote as Ω the discourse universe. For an open formula φ containing the free variable z, the set A of individuals of Ω totally satisfying the formula φ is called the *set referred* by φ.[2] In other words, for a valuation s, $\mathfrak{A} \models^{s(0/a)}_M \varphi(z) \Leftrightarrow a \in A$. This can be characterized by a symbolic proportion Q_α of elements of Ω totally satisfying φ with respect to uniform probability

[2] By using the concept of *placeholder variables* in lambda-abstraction, Bacchus [3] considers that a Boolean open formula can refer to the set of all instances of its free variables, specified as placeholders, satisfying the formula.

distribution on Ω. So, Q_α can be considered as the *symbolic degree of statistical probability* of the set A in Ω, i.e., the frequency or the absolute proportion of elements of Ω which are in A. Linguistically speaking, this can receive the following translation: "A proportion Q_α of individuals of Ω are in A", which is classically denoted "Q_α Ω's are A's". Then, the managing of statistical probabilities can be handled by enriching the syntax and the semantic of our M-valued logic by adding a particular predicate, denoted as **Prop**, and a formation rule based on this predicate, and the axioms governing the use of formulae referring to **Prop**. Of course, in order to obtain a theory of symbolic statistical probabilities, these axioms may be justified at a metalogical level.

In the following, C_1 denotes the set of formulae φ such that, for any interpretation $\mathfrak{A}$ and any z in the domain, $\varphi(z)$ is totally true or totally false in $\mathfrak{A}$: $C_1 = \{\varphi \in \mathfrak{F} \mid \forall a \in \Omega, \mathfrak{A} \models_M^{s(0/a)} \varphi(z)$ or $\mathfrak{A} \models_1^{s(0/a)} \varphi(z)\}$. We can now put the following definition.

Definition 2. The predicate **Prop** is formally introduced as follows:

- If $\phi \in C_1$, then $\mathbf{Prop}(\phi) \in \mathfrak{F}$.
- Any interpretation $\mathfrak{A}$ associates with the predicate **Prop** the multiset $\mathfrak{S}$.
- Hence, for any $\phi \in C_1$: $\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi) \Leftrightarrow \langle\phi\rangle \in_\alpha \mathfrak{S} \Leftrightarrow \mathbf{Prop}(\phi)$ is τ_α-*true in*-$\mathfrak{A}$.[3]

As noted before, this definition can receive the following translation: "Q_α Ω's are A's" (i.e., a proportion Q_α of individuals of Ω are in A). So, it is possible to associate with any $\tau_\alpha \in \mathfrak{L}_M$ a vague proportional quantifier denoted Q_α. In the following, we denote as $\mathfrak{Q}_M = \{Q_\alpha,\ \alpha \in \mathfrak{M}\}$ the resulting set of proportional quantifiers.

Choice of quantifiers. By choosing $M = 7$, we can introduce: $\mathfrak{Q}_7 = \{$none, very-few (or almost-none), few, about half, most, almost-all, all$\}$ that corresponds to the symbolic degrees of statistical probability.

The previous definition leads to the following one.

Definition 3. Let $\mathfrak{Q}_M$ be the set of the vague proportional quantifiers: $\mathfrak{Q}_M = \{Q_\alpha, \alpha \in [1, M]\}$. Then, "$\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi)$" will mean that "$Q_\alpha$ individuals of Ω totally satisfy ϕ in-$\mathfrak{A}$", if and only if, the subset referred by ϕ belongs to the multiset $\mathfrak{S}$ with a degree τ_α.

Definitions 2 and 3 refer to the two different values Q_α and τ_α, even if a one-to-one correspondence associates the corresponding graduations scales. The difference results from the fact that they are semantically different: τ_α represents a truth degree and Q_α stands for the proportion expressed in the quantified assertion. Let us take basic examples to explain the notions described in definitions 2 and 3.

First, the definition 2 allows us to define the proportion of elements of Ω which satisfy ϕ. Hence, if we have, for example, $\mathfrak{A} \models_2 \mathbf{Prop}(\text{Fly}(z))$, this means that the predicate **Prop** is verified at the level "very-little" when dealing with Fly. This allows

[3] If no confusion is possible, $\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi)$ stands for $\mathfrak{A} \models_\alpha^{s(0/\phi)} \mathbf{Prop}(\varphi)$.

to represent that the proportion of elements of Ω which belongs to the sets of elements which fly is "very-few".

To illustrate now the definition 3, we can give some examples:

1. Let us consider a city V and its residents. So, the set Ω corresponds to the set of the residents of the city. If we consider the quantified assertion "Most residents of the city V are young", ϕ is the formula referring to the set of elements of Ω being young and **Prop**(ϕ) is "very-true-in-$\mathfrak{A}$" ("very-true" is the linguistic truth degree τ_α of **Prop**(ϕ)). The quantified assertion is also interpreted as "Q_α individuals of Ω satisfy ϕ in-$\mathfrak{A}$" where Q_α is the quantifier "Most".

2. Let us now take the example of the integers between 1 and 9. In this set, since 1, 3, 5, 7 and 9 are odd and the other integers are not odd, we can say that "About half numbers are odd". Here, the set Ω is the set of integers $[1, 9]$, the formula ϕ refers to the set of odd numbers, τ_α is the truth degree "moderately-true" of **Prop**(ϕ) and Q_α is the quantifier "About half".

3. If we take the quantified assertion "Almost all students are single", we choose Ω to be the set of the students, "single" is the set associated to the formula ϕ, "almost-true" is the truth degree τ_α and "almost all" is the quantifier Q_α.

Let us now consider quantified assertions, like "Most birds fly", classically denoted in the following form "Q A's are B's". The idea proposed in Pacholczyk [27] for the representation of "Q A's are B's" was to interpret it as a symbolic conditional uncertainty of the event B given A. Our interpretation will be in terms of the *symbolic relative (or conditional) proportion of individuals of B in A*. Therefore, as in [27,31], we can generalize the classical definition of conditional statistical probability in a symbolic context, by using a "symbolic probabilistic division" operator, denoted as C, or equivalently, a "symbolic probabilistic multiplication" operator, denoted as I. These two operators have been defined in [27]. The operator I is an application of $\mathfrak{Q}_M^2$ into $\mathfrak{Q}_M$, that verifies the classical properties of the probabilistic multiplication (commutativity, absorbent element: Q_1, neutral element: Q_M, monotony, associativity, idempotence: Q_2). The operator C is an application of $\mathfrak{Q}_M^2$ into $\mathfrak{P}(\mathfrak{Q}_M)$, which is the set of parts of $\mathfrak{Q}_M$. C is deduced from I by a unique way as follows: $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda) \Leftrightarrow Q_\lambda = \mathrm{I}(Q_\alpha, Q_\mu)$. Among the different tables of the operator C which verify the axioms chosen in [27], in $\mathfrak{L}_7$ we have chosen table 1 presented in appendix A. The corresponding operator I is defined in appendix A by table 2.[4]

By using previous definition of absolute statistical probability, we can define the notion of conditional statistical probability of ψ given ϕ, denoted as $\mathfrak{A} \models_\mu$ **Prop**($\psi|\phi$), which can be viewed a a symbolic generalisation of conditional probability in classical probability theory.

[4] It can be noted that the operators $\cup$, $\cap$, $\neg$, $\supset$ have been defined in the M-valued logic in order to deal with partial truth degrees of imprecise (or fuzzy) knowledge. But, they do not allow us to govern the particular predicate **Prop**, since statistical probabilities are not truth-functional, in this sense that the statistical probability of a compound formula is not a function of the statistical probabilities of its parts. In order to propose a symbolic representation of the statistical probability, we have to define the necessary operators governing this symbolic concept.

Definition 4. Let ϕ and ψ be formulas of C_1, the symbolic conditional statistical probability of ψ given ϕ, denoted as $\mathfrak{A} \models_\mu \mathbf{Prop}(\psi|\phi)$, is defined by the symbolic division of the symbolic degree of $\mathbf{Prop}(\psi \cap \phi)$ by that of $\mathbf{Prop}(\phi)$ as follows: $\{\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi), \mathfrak{A} \models_\lambda \mathbf{Prop}(\psi \cap \phi)\} \Rightarrow \mathfrak{A} \models_\mu \mathbf{Prop}(\psi|\phi)$ with $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda)$.

An alternative formulation of the above approach to conditional statistical probabilities may be carried out as follows. It appears that Q_μ may be seen as the "measure" of the extend to which the set of individuals of Ω which totally satisfy ψ in-$\mathfrak{A}$ is included in the set of individuals which totally satisfy ϕ in-$\mathfrak{A}$. So, we can introduce the notion of *partial inclusion* of a set in another, like in [2]. If the formulas ϕ and ψ refer respectively to the subsets A and B of Ω, the symbolic degree of partial inclusion of A in B, coincides with the symbolic degree of conditional statistical probability of individuals of the second in the first, with respect to uniform probability distribution on Ω. If we suppose that ϕ and ψ refer respectively to subsets A and B of Ω in the interpretation $\mathfrak{A}$, previous equivalence leads to the following definition of partial inclusion: "$A \subset_\mu B$" $\Leftrightarrow$ "$\mathfrak{A} \models_\mu \mathbf{Prop}(\psi|\phi)$" $\Leftrightarrow$ Q_μ A's are B's. Moreover, T being a tautology we have: $\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi|\mathsf{T}) \Leftrightarrow \mathfrak{A} \models_\alpha \mathbf{Prop}(\phi)$. In other words, absolute probability appears as a particular case of conditional probability: "$\Omega \subset_\alpha A$" $\Leftrightarrow$ "$\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi)$". Thus, previous definition can be rewritten in terms of set inclusion as follows.

Definition 5. Given an interpretation $\mathfrak{A}$, let us suppose that the formulas ϕ and ψ refer respectively to the subsets A and B of Ω. Then, in terms of *partial inclusion* in the set theory, $A \subset_\mu B$ if and only if $\mathfrak{A} \models_\mu \mathbf{Prop}(\psi|\phi)$. Then, "$A$ is included into B with a degree Q_μ" is equivalent to say that "Among the individuals of Ω which totally satisfy ϕ in-$\mathfrak{A}$, Q_μ totally satisfy ψ in-$\mathfrak{A}$". Linguistically speaking, this will be denoted as "Q_μ A's are B". In other words, the following assertions are equivalent:

- if $\{\mathfrak{A} \models_\alpha \mathbf{Prop}(\phi)$ and $\mathfrak{A} \models_\lambda \mathbf{Prop}(\psi \cap \phi)\}$, then $\mathfrak{A} \models_\mu \mathbf{Prop}(\psi|\phi)$ with $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda)$,
- if $\{\Omega \subset_\alpha A,\ \Omega \subset_\lambda A \cap B\}$, then $A \subset_\mu B$ with $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda)$.

Remark 1. It appears that: "$\{\Omega \subset_\alpha A,\ \Omega \subset_\lambda A \cap B\} \Rightarrow A \subset_\mu B$, with $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda)$" be viewed as a symbolic generalisation of the classical property: $\mathbf{Prop}(B|A) = \mathbf{Prop}(A \cap B)/\mathbf{Prop}|(A) = |A \cap B|/|A|$.

Let A and B be subsets of Ω. It is easy to prove the following properties.

Proposition 1. $A \subset_M B \Leftrightarrow A \subset B \Leftrightarrow A \cap B = A$.

Since the degree Q_M corresponds with the quantifier "All", then $\subset_M$ coincides with the inclusion in set theory.

Proposition 2. $A \subset_1 B \Leftrightarrow A \cap B = \emptyset$.

Since the degree Q_1 corresponds with the quantifier “None”, this means that A and B are disjoint.

Example 1. By using $\mathfrak{L}_7$, let us suppose that the domain of discourse consists residents of the city V. Knowing that: “Most residents of the city V are young” and “Half of residents of the city V are young single”. These assertions are respectively translated in our model by: $\Omega \subset_5$ Young and $\Omega \subset_4$ Young $\cap$ Single. Definition 5 gives us: Young $\subset_\mu$ Single with $Q_\mu \in \mathrm{C}(Q_5, Q_4) = \{Q_5\}$. Then we obtain: “Most young people are single”.

4. Axiomatic of symbolic statistical probabilities

We can now put the axioms governing the concept of symbolic statistical probabilities. The axioms are expressed in terms of partial inclusion. Each of them is justified at a metalogical level. In the following, A and B denote subsets of Ω.

Axiom 1. $A \cap B \neq A,\ \Omega \subset_\alpha A,\ \Omega \subset_\alpha A \cap B$ and $Q_\alpha \in [Q_3, Q_{M-1}] \Rightarrow A \subset_{M-1} B$ (axiom defining “Almost-all”).

Qualitatively the subsets A and $A \cap B$ can have the same symbolic degree of proportions without being equal. This is the case, when $A \cap B$ is equal to the set A without one or some individuals. This can qualitatively be translated by saying that “A and $A \cap B$ are almost equal” or “Almost all A’s are B’s”. This is not always the case when the proportion of A is very weak (associated with $Q_2 =$ *Very-few*).

Axiom 2. $\Omega \subset_\alpha A,\ Q_\alpha \in [Q_2 Q_{M-1}]$ and $A \subset_{M-1} B \Rightarrow \Omega \subset_\alpha A \cap B$ (axiom defining “Almost-all”).

When we have “Almost all A’s are B’s”, we know that $A \neq A \cap B$, but we can say that A and $A \cap B$ are almost equal and therefore A and $A \cap B$ have the same symbolic degree of proportions.

Axiom 3. $\Omega \subset_\alpha A \Leftrightarrow \Omega \subset_{n(\alpha)} \overline{A}$ with $n(\alpha) = M + 1 - \alpha$ (axiom defining the dual quantifier).

Generally the dual quantifier of Q_α corresponds with $Q_{n(\alpha)}$ (“Few” is the dual quantifier of “Most”).

Axiom 4. $\Omega \subset_\alpha A,\ \Omega \subset_\beta B,\ A \cup B \neq \Omega$ and $A \cap B = \emptyset \Rightarrow \Omega \subset_r A \cup B$ with $Q_r \in S(Q_\alpha, Q_\beta)$ (axiom defining the symbolic proportion of disjoint sets union).

Classically, when A and B are disjoint, the absolute proportion of their union is the sum of their absolute. If the union A and B is different from Ω (otherwise, the symbolic proportion degree of their union is evidently Q_M) and that they are disjoint, then the symbolic proportion degree of their union belongs to the "*symbolic sum*" of their symbolic proportion degrees. The symbolic sum denoted S is introduced in a way that it gives an interval containing one or two values. The lower bound of this interval is greater than or equal to each symbolic value of two arguments of S. Since the set $A \cup B$ is different from Ω, the maximal degree that can take the upper bound of the interval is Q_{M-1}. The use of an interval rather than a single degree is due to the degree Q_2. It is justified by the fact that the addition of one or some elements (i.e., a very weak quantity) to a set can either preserve its symbolic degree of proportion or increase it at most one degree.

Definition 6. The symbolic addition S is a commutative application of $\mathfrak{Q}_M^2$ into $\mathfrak{P}(\mathfrak{Q}_M)$. By supposing that $\alpha + \beta \leqslant M + 1$, S is defined as follows:

$$S(Q_\alpha, Q_\beta) = \begin{cases} \{Q_\alpha\} & \text{if } \beta = 1, \\ [Q_{\alpha+\beta-2}, Q_{\alpha+\beta-1}] & \text{if } \alpha \neq 1,\ \beta \neq 1,\ \alpha+\beta \leqslant M, \\ \{Q_{M-1}\} & \text{if } \alpha + \beta = M + 1. \end{cases}$$

In agreement with axiom 3, it is necessary to have $\alpha + \beta \leqslant M + 1$. Indeed, $A \cap B = \emptyset$ implies that $B \subset \overline{A}$. Now axiom 3 gives $\Omega \subset_{n(\alpha)} \overline{A}$. Intuitively $\beta \leqslant n(\alpha)$ (for $B \subset \overline{A}$) therefore, $\alpha + \beta \leqslant \alpha + n(\alpha) = M + 1$. Defining Inf and Sup as two applications of $\mathfrak{Q}_M^2$ into $\mathfrak{Q}_M$, we obtain respectively the lower bound and the upper bound of an interval of Q_M so we can write: $\mathrm{S}(Q_\alpha, Q_\beta) = [\mathrm{Inf} \circ \mathrm{S}(Q_\alpha, Q_\beta), \mathrm{Sup} \circ \mathrm{S}(Q_\alpha, Q_\beta)]$ or, more simply, $[\mathrm{Inf\,S}(Q_\alpha, Q_\beta), \mathrm{Sup\,S}(Q_\alpha, Q_\beta)]$. We can prove that the applications Inf S and Sup S verify the properties of a T-conorm (neutral element, commutativity, monotony properties and associativity).

Definition 7. Given S, we can define the "symbolic subtraction" denoted D as an application of $\mathfrak{Q}_M^2$ into $\mathfrak{P}(\mathfrak{Q}_M)$ such that: if $Q_r \in \mathrm{S}(Q_\alpha, Q_\beta)$, then $Q_\beta \in \mathrm{D}(Q_r, Q_\alpha)$ and $Q_\alpha \in \mathrm{D}(Q_r, Q_\beta)$. Then D can be deduced from S:

$$\mathrm{D}(Q_r, Q_\beta) = \begin{cases} \{Q_r\} & \text{if } \beta = 1, \\ \{Q_2\} & \text{if } r = \beta \in [2, M-1], \\ [Q_{r+1-\beta}, Q_{r+2-\beta}] & \text{if } 2 \leqslant \beta < r \leqslant M - 1. \end{cases}$$

Remark 2. In this paper, for $M = 7$ we obtain the operators S and D defined by tables 3 and 4 (see appendix A).

5. Fundamental properties

Let A and B be subsets of Ω. The following properties can be viewed as symbolic generalization properties of classical statistical probabilities. The proofs of the properties can be found in appendix B.

Proposition 3. If $\Omega \subset_\alpha A$ and $A \subset B$, then $\Omega \subset_\beta B$ with $Q_\alpha \leqslant Q_\beta$.

Proposition 3 shows that the symbolic degree of proportion of a set is greater than or equal to its subsets. Classically, the proportion of a set is strictly greater than one of its strict subsets, while qualitatively, a set and one of its subsets can have the same symbolic degree of proportion (cf. axiom 1).

Proposition 4. If $\Omega \subset_\alpha A$, $\Omega \subset_\lambda A \cap B$ and $A \neq \Omega$, then $\Omega \subset_\gamma A \setminus B$ with $Q_\gamma \in \mathrm{D}(Q_\alpha, Q_\lambda)$.

Proposition 4 generalizes the property: $|A \setminus B|/|\Omega| = (|A| - |A \cap B|)/|\Omega|$.

Proposition 5. If $\Omega \subset_\alpha A$, $\Omega \subset_\beta B$, $\Omega \subset_\lambda A \cap B$ and $A \cup B \neq \Omega$, then $\Omega \subset_r A \cup B$ with $Q_r \in \mathrm{U}(Q_\alpha, Q_\beta, Q_\lambda)$ where $\mathrm{U}(Q_\alpha, Q_\beta, Q_\lambda) = [\mathrm{Inf\,S}(Q_\alpha, \mathrm{Inf\,D}(Q_\beta, Q_\lambda)), \mathrm{Sup\,S}(Q_\alpha, \mathrm{Sup\,D}(Q_\beta, Q_\lambda))]$ if $\alpha + \beta - \lambda \leqslant M - 1$, and $\mathrm{U}(Q_\alpha, Q_\beta, Q_\lambda) = \{Q_{M-1}\}$ if $\alpha + \beta - \lambda = M$.[5]

Corollary 6. If $\Omega \subset_\alpha A$, $\Omega \subset_\beta B$, $\Omega \subset_r A \cup B$ and $A \cup B \neq \Omega$, then $\Omega \subset_\lambda A \cap B$ with $Q_\lambda = Q_2$ if $\alpha + \beta - r = 1$ and $Q_\lambda \in [\mathrm{Inf\,D}(Q_\beta, \mathrm{Sup\,D}(Q_r, Q_\alpha)), \mathrm{Inf}\{\mathrm{Sup\,D}(Q_\beta, \mathrm{Inf\,D}(Q_r, Q_\alpha)), Q_\alpha, Q_\beta\}]$ otherwise.

Proposition 5 and corollary 6 generalize the classical property:

$$|A \cup B|/|\Omega| = \big(|A| + |B| - |A \cap B|\big)/|\Omega|.$$

6. Inference with quantifiers

Reasoning on quantifiers is called by Zadeh [35] *syllogistic reasoning*, where a syllogism is an inference rule that consists in deducing a new quantified statement from one or several quantified statements. As an inference scheme, a *syllogism* may generally be expressed in the form:

$$\frac{\begin{array}{c} Q_{\mu 1}\ A\text{'s are } B\text{'s} \\ Q_{\mu 2}\ C\text{'s are } D\text{'s} \end{array}}{Q_\mu\ E\text{'s are } F\text{'s with } Q_\mu \in [Q_x, Q_y] \subseteq [Q_1, Q_M]}$$

where E and F are sets resulting from application of set operators on A, B, C or D, and bounds Q_x and Q_y are in accordance with $Q_{\mu 1}$ or $Q_{\mu 2}$. The quantifier "All" is represented by the implication using the quantifier $\forall$ in classical logic or by the inclusion in set theory. The classical implication and the inclusion propagate inferences by transitivity, contraposition, disjunction or by conjunction. From one or several statements

[5] Inf D and Sup D are defined like Inf S and Sup S.

quantified by "All", these inferences enable to generate new statements likely quantified by "All". Nevertheless, most of these inferences are not valid for other quantifiers, i.e., for $Q_\mu \in [Q_2, Q_{M-1}]$. For example, from "*Most A's are B's*" and "*Most B's are C's*" one cannot always have "*Most A's are C's*". That is due to the fact that the inference by transitivity is not valid for the quantifier "Most". The invalid inference has been considered as a case of *total ignorance*.

6.1. Valid inferences with quantifiers

We consider that an inference is valid, if we deduce $Q_\mu \in [Q_x, Q_y]$,[6] with Q_x or Q_y is in accordance with $Q_{\mu 1}$ or $Q_{\mu 2}$. We present some valid inferences. Each of them is illustrated by an example.

Proposition 7 (Relative Duality).

$Q_{\mu 1}$ A's are B's

$Q_{\mu 2}$ A's are $A \setminus B$ and $Q_{\mu 2}$ A's are $\overline{B}$'s with $Q_{\mu 2} = Q_{n(\mu 1)}$ if $Q_{\mu 1} \neq Q_{n(\mu 1)}$ and $Q_{\mu 2} \in [Q_{n(\mu 1)}, Q_{n(\mu 1)+1}]$ otherwise.

Example 2.

Almost all students are unmarried

Very few students are married.

Proposition 8 (Mixed Transitivity).

$Q_{\mu 1}$ A's are B's
All B's are C's

$Q_{\mu 2}$ A's are C's with $Q_{\mu 1} \leqslant Q_{\mu 2}$.

Example 3.

Most students are young (less than 25 years)
All young people are nonretired

At least most students are nonretired.

[6] Let us point out that $[Q_x, Q_y]$ denoted a subinterval of the set of the vague proportional quantifiers $\mathfrak{Q}_M = \{Q_\alpha,\ \alpha \in [1, M]\}$, and $Q_\mu \in [Q_x, Q_y]$ denotes a value of this subinterval (but not a new value of this subinterval). Then, $Q_\mu \in [Q_x, Q_y]$ means that the symbolic value Q_μ is a value of $\mathfrak{Q}_M$ bounded by Q_x and Q_y.

Proposition 9 (Exception).

Q_μ A's are B's
All C's are A's
All C's are $\overline{B}$

Q_γ A's are $\overline{C}$, with $Q_\gamma \in [Q_\mu, Q_{M-1}]$.

Example 4.

Most birds fly
All ostriches are birds
All ostriches do not fly

Most or almost all birds are not ostriches.

Proposition 10 (Union Right).

$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ A's are C's

Q_μ A's are $(B \cup C)$'s, with $Q_\mu \in [Q_{\mathrm{Max}(\mu 1, \mu 2)}, Q_{M-1}]$.

Example 5.

Most students are single
Very few students are taxable

Most or almost all students are single or taxable.

Proposition 11 (Intersection Right).

$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ A's are C's

Q_μ A's are $(B \cap C)$'s, with $Q_\mu \leqslant Q_{\mathrm{Min}(\mu 1, \mu 2)}$.

Example 6.

Few salaried people are official
Most salaried people are taxable

At least most salaried people are taxable official.

Proposition 12 (Mixed Union Left).

Q_μ A's are C's
All B's are C's

Q_γ $(A \cup C)$'s are B's with $Q_\gamma \in [Q_\mu, Q_{M-1}]$.

Example 7.

Most young people are single
All the catholic priests are single

Most or almost all young people or catholic priests are single.

Proposition 13 (Intersection/Product Syllogism).

$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ $(A \cap B)$'s are C's

Q_{μ} A's are $(B \cap C)$'s, with $Q_{\mu} = \mathrm{I}(Q_{\mu 1}, Q_{\mu 2})$.

Example 8.

Most students are young
Almost all young students are unmarried

Most students are young and unmarried.

Proposition 14 (Contraction).

$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ $(A \cap B)$'s are C's

Q_{μ} A's are C's, with $Q_{\mu} = [\mathrm{I}(Q_{\mu 1}, Q_{\mu 2}), Q_{\mathrm{Max}(M-1,\mu 2)}]$.

Example 9.

Most students are young
Almost all young students are unmarried

Most or almost all students are young and unmarried.

Proposition 15 (Intersection/Quotient syllogism).

$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ A's are C's
$Q_{\mu 3}$ $(A \cap B)$'s are C's

Q_{μ} $(A \cap C)$'s are B's, with $Q_{\mu} \in \mathrm{C}(Q_{\mu 2}, \mathrm{I}(Q_{\mu 1}, Q_{\mu 3}))$.

Example 10.

Most students are young
Most students nonsalaried
Almost all young students are nonsalaried

Almost all nonsalaried students are young.

Proposition 16 (Weak Transitivity).

All B's are A's
$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ B's are C's

Q_μ A's are C's with $Q_\mu \in [\mathrm{I}(Q_{\mu 1}, Q_{\mu 2}), Q_{M-1}]$.

Example 11.

All salaried people are active
Most active people are salaried
Most salaried people are taxable

Q_μ active people are taxable with $Q_\mu \in$ [half, almost all].[7]

6.2. *Valid inferences with the quantifier "almost-all"*

We present three inferences only valid with the quantifier "Almost-all". They result from the axioms of quantifier "Almost-all" (cf. axiom 1, axiom 2). These inferences can be viewed as counterparts[8] of inference rules of Adams [1], Pearl [28] and Bacchus et al. [4], where "Almost all" is interpreted as proportion arbitrarily infinitesimal close to 1. Our symbolic approach leads to the following results which are in accordance with the previous ones, since we obtain a proportion Q_μ belonging to a subinterval of $\mathfrak{Q}_M$ containing the value "Almost all". It is clear that the meaning associated with the quantifier "Almost all" in our approach, as in the ones of Adams and Pearl, differs from the meaning that it receives in natural language. Indeed, a speaker does not refer to infinitesimal proportion close to 1, since linguistically speaking, "Almost all" is only included in "Most". As noted before, in our approach, "Most" and "Almost all" define two different values of proportions of $\mathfrak{Q}_M$, "Most" being less than "Almost all", which is very close to "All", but not included in "Most".

Corollary 17 (Contraction).

Almost-all A's are B's
Almost-all $(A \cap B)$'s are C's

Almost-all A's are C's.

Example 12.

Almost all students are young
Almost all young students are single

Almost all students are single.

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[7] Linguistically speaking, the result $Q_\mu \in$ [half, almost all], which means that Q_μ receives one of the three possible values, can be rewritten in the following linguistic form: either "half" or "most" or "almost all".

[8] Pearl's approach is introduced for default reasoning, then his inferences are not exactly syllogisms, but they are rather non-monotonic inferences about particular individuals from defaults.

Remark 3. Clearly, "Almost all A's are B's" means that the proportion of elements of A being elements of B is very very important. In other words, the proportion of A's not being B's is very very weak. Then, for the corollary of contraction, since the proportion of A's being not B's and the proportion of A's and B's being not C's are very very weak (due to the use of infinitesimal proportion), it is clear that we obtain the proportion of A's being not C's is also very very weak (i.e., "Almost all A's are C's").

Proposition 18 (Cumulativity).

Almost-all A's are B's
Almost-all A's are C's.

Q_μ $(A \cap B)$'s are C's, with $Q_\mu \in$ [Most, All].

Example 13.

Almost all students are young
Almost all students are single

At least most young students are single.

Proposition 19 (Union Left).

Almost-all A's are C's
Almost-all B's are C's.

Q_μ $(A \cup B)$'s are C's, with $Q_\mu \in$ [Most, Almost-all].

Example 14.

Almost all students are single
Almost all priests are single

Most or almost all students or priests are single.

6.3. Monotonic aspect of reasoning with quantifiers

We can note that the reasoning with quantifiers is monotonic [21] in the following sense: when a knowledge base contains: $Q_{\mu 1}$ A'*s are* B'*s* and $Q_{\mu 2}$ C'*s are* D'*s*, when we deduce Q_μ *E's are F's* with $Q_\mu \in [Q_x, Q_y]$, and if one adds in the base or one deduces by an other inference new information: $Q_{\mu'}$ *E's are F's* with $Q_{\mu'} \in [Q_{x'}, Q_{y'}]$, then one must have $[Q_x, Q_y] \cap [Q_{x'}, Q_{y'}] \neq \emptyset$ and, finally, one will have: $Q_{\mu'}$ *E's are F's* with $Q_{\mu'} \in [Q_x, Q_y] \cap [Q_{x'}, Q_{y'}]$. In other words, the new knowledge can only tighten the interval $[Q_x, Q_y]$, that maintains the coherence between the quantified statements. There is an *inconsistency*, if $[Q_x, Q_y] \cap [Q_{x'}, Q_{y'}] = \emptyset$.

Example 15. The following example proposed by Loui [22] among benchmarks problems contains the quantified statements:
S1: Most dancers are not ballerinas.

S2: Most dancers are graceful.
S3: Most graceful dancers are ballerinas.
S1 $\Rightarrow$ "Few dancers are ballerinas" (Relative duality).
S1 and S2 $\Rightarrow$ "Either half or most or almost all dancers are ballerinas" (Contraction).
Since {Few} $\cap$ [Half, Almost-all] $= \emptyset \Rightarrow$ three statements S1, S2 and S3 are inconsistent.

It is interesting to focus on when such an inconsistency can occur. Let us notice that it is clear that, when a set of quantified assertions is inconsistent, it does not exist a description of the world in terms of proportions that verifies the set of quantified assertions. As the quantified assertions that are used in our paper come from statistical probabilities (proportions), it is true to say that such an inconsistency can not occur.

An inconsistent set of quantified assertions denotes subjective probabilities but it does not have any significance in our model.

However, in our symbolic model, the assertions are treated in a qualitative level that is the probabilities are not numerical but give a rank in restricted scale of values. Since the values are not numerical, it is necessary that an agent expresses the qualitative statistical values. This qualitative data becomes then subjective and are an estimation of the proportions. Hence, since the assertions are expressed by an agent, they can be inconsistent in the sense we give. Note that this is not due to the syllogisms since they use symbolic operators (I, C, S) that do not contradict the properties of numerical operators (even if they are defined in a subjective way).

The given example has been proposed for default reasoning [22] where the conflict occurs at the instanciation level. But we deal in this part with assertions about classes of individuals. So, we can say that if the agent expresses assertions based on a qualitative estimation of proportions, the determination of inconsistency allows to determine if the assertions really correspond to qualitative statistical values.

7. Comparison with Bacchus's approach

We can justify the correctness of our approach to linguistic quantification and the soundness of our results. The basic notions defining (1) the representation of linguistic modifiers, and (2) the deductive process dealing with quantified assertions results from the papers of Bacchus [3]. As noted before, Pacholczyk's approach is more devoted to uncertainty management (section 3) and will not be compared with our current work. But, it is interesting to compare our model with the ones of Bacchus. There are two levels where we can compare our approach to the one proposed by Bacchus. The first level concerns the *representation* of statistical quantified assertions and the second level deals with *syllogistic reasoning*.

7.1. Representation

At the representational level, Bacchus extends first-order classical logic by introducing a new operator (denoted by [ ]) in order to define numerical proportions. In our approach, unless to add a new operator, we have introduced in our M-valued symbolic logic a new predicate (that can be viewed as an element of a meta-logic) and we have given the axioms governing it. This can be interpreted (see paragraphs 3–5) as a symbolic generalization of classical absolute and conditional statistical probabilities (proportions).

About the quantifiers, Bacchus' framework is defined upon statistical assertions using numerical values but it is only used for the symbolic values "majority" (which is interpreted as a proportion > 0.5) and "minority". Bacchus call his quantifier expressing the majority "most". So, he only focuses on one linguistic quantifier (and its dual one) defining typicality. In our work, the aim is different since we do not want to represent the notion of majority but to capture the whole set of symbolic proportions. That's why we use several different quantifiers (seven in this paper) which describe a scale of quantifiers. Clearly, this scale defines a set of symbolic quantifiers which can be seen as a refinement of the symbolic quantifiers used by Bacchus.

7.2. Syllogistic reasoning

If we focus now on the syllogistic reasoning, we can verify that our approach leads to find similar syllogisms as the ones that can be found with Bacchus' proposal.

If Q is associated with a numerical value (or a numerical interval $[a, b]$) then in Bacchus' approach, we can give the following syllogisms:

1. *Mixed Transitivity*.

Q A's are B's
1 B's are C's (1 is equivalent to 100% or "All")

$[Q, 1]$ A's are C's (i.e., Q' A's are C's with $Q' \geqslant Q$).

Our approach gives the same result (section 6, proposition 8):

$Q_{\mu 1}$ A's are B's
All B's are C's

$Q_{\mu 2}$ A's are C*'s with* $Q_{\mu 2} \geqslant Q_{\mu 1}$.

2. *Intersection/Product syllogism*

$Q1$ A's are B's
$Q2$ $(A \cap B)$'s are C's

$Q1 * Q2$ A's are $(B \cap C)$'s

(where * stands for the multiplication operator).

Our approach gives a similar result (section 6, proposition 13), since the operator I stands for an operator having in $\mathfrak{L}_M$ the properties of a multiplication operator [27].

$$\frac{\begin{array}{l} Q_{\mu 1}\ A\text{'s are } B\text{'s} \\ Q_{\mu 2}\ (A \cap B)\text{'s are } C\text{'s} \end{array}}{Q_{\mu}\ A\text{'s are } (B \cap C)\text{'s, with } Q_{\mu} = \mathrm{I}(Q_{\mu 1}, Q_{\mu 2}).}$$

It is easy to verify that propositions 9–12, and 14–16, of section 6 lead to similar results.

It is clear that they correspond to the same syllogisms since, for each syllogism, the resulting assertion is the same and the quantifier is obtained is the same way in the numerical and in the symbolic setting (that is using the same combination of the operators).

Moreover, the operators C (division), I (product), S (addition), D (difference) are the symbolic counterparts of the four classical operators (see appendix A). The operators defined for the symbolic setting respect the properties of classical operator such as, depending on the considered operator, associativity, existence of a neutral element, commutativity, monotony, ...

The behavior of our syllogistic reasoning when dealing with precise values is then depending on the symbolic operators used for the syllogism. The question is to verify that they are in accordance with the classical operators used in a numerical setting.

The problem is that it is not possible to prove that the symbolic operators are in total accordance with numerical operators. Indeed, it does not exist an isomorphism between the numerical and the symbolic settings (as shown, for example, by Kaufmann [12]). So, it is not possible to give an interface between numerical and symbolic quantifiers allowing to compare the behaviors of the two different systems.

Finally, let us say that, on one hand, Bacchus' proposal is adapted when the given data are expressed with precise values but it would not be suitable when the information is symbolic. On the other hand, our work is useful when there do not exist precise values but when the information are only expressed in terms of symbolic values (which is suitable with the initial aim of our work).

8. Conclusion and extensions

In this paper, we have presented a symbolic approach to quantifiers used in the natural language to express a qualitative evaluation of proportions. This approach is basically devoted to reason qualitatively on *quantified assertions*, since we provide inference rules based upon statements involving *linguistic quantifiers*. These deduction rules allow us to derive new assertions from the initial ones. All the deductions are proved to be true in the semantic model dealing with statistical conditional probabilities. We can point out that the treated examples lead to results in good accordance with the ones resulting from the common sense reasoning, and this within a qualitative context. Moreover, it appears that our approach gives us similar syllogisms as the ones obtained with Bacchus' proposal.

Another development of our work (which is not presented in this article), concerns the management of *particular individuals*, and this, by using knowledge based upon quantified assertions and facts. To this aim, we propose a symbolic model based upon a *direct inference principle* and a choice of the appropriated *reference class* [3,20,29, 30]. More precisely, reasoning about particular individuals is based upon quantified assertions in the sense that the subjective probability associated with a property about a particular individual is derived from the proportion of individuals verifying this property (for example, knowing that "Most smokers may have lung cancer", a doctor thinks that it is very-probable that Martin which is a smoker will have a lung cancer). Contrary to syllogistic reasoning which is a monotonic process, reasoning on particular individuals constitutes a non monotonic reasoning process. So, it will be interesting to verify that the properties associated with this process fullfil the basical postulates of a non monotonic relation, like the ones defining System P [19]. This point is actually on study.

A future extension of our framework should be the treatment of implicative information. Our work seems to extend the space in which probabilistic and statistical characteristics of uncertain implications can be described and processed in terms of logical syllogisms. Indeed, each quantified assertion "Q_α A's are B's", where Q_α is interpreted as a conditional probability, can also be viewed as a graduated implication $Q_\alpha\ (A \rightarrow B)$.

Appendix A. Tables of operators C, I, S and D

Remark 4. In the following tables, $Q_{a,b}$ stands for interval $[Q_a, Q_b]$.

Table 1
Operator C.

C	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
Q_1	$Q_{1,7}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Q_2	$\{Q_1\}$	$Q_{2,7}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Q_3	$\{Q_1\}$	$Q_{2,5}$	$Q_{6,7}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Q_4	$\{Q_1\}$	$Q_{2,4}$	$\{Q_5\}$	$Q_{6,7}$	$\emptyset$	$\emptyset$	$\emptyset$
Q_5	$\{Q_1\}$	$Q_{2,3}$	$\{Q_4\}$	$\{Q_5\}$	$Q_{6,7}$	$\emptyset$	$\emptyset$
Q_6	$\{Q_1\}$	$\{Q_2\}$	$\{Q_3\}$	$\{Q_4\}$	$\{Q_5\}$	$Q_{6,7}$	$\emptyset$
Q_7	$\{Q_1\}$	$\{Q_2\}$	$\{Q_3\}$	$\{Q_4\}$	$\{Q_5\}$	$\{Q_6\}$	$\{Q_7\}$

Table 2
Operator I.

I	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1
Q_2	Q_1	Q_2	Q_2	Q_2	Q_2	Q_2	Q_2
Q_3	Q_1	Q_2	Q_2	Q_2	Q_2	Q_3	Q_3
Q_4	Q_1	Q_2	Q_2	Q_2	Q_3	Q_4	Q_4
Q_5	Q_1	Q_2	Q_2	Q_3	Q_4	Q_5	Q_5
Q_6	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_6
Q_7	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7

Table 3
Operator S.

S	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
Q_1	$\{Q_1\}$	$\{Q_2\}$	$\{Q_3\}$	$\{Q_4\}$	$\{Q_5\}$	$\{Q_6\}$
Q_2	$\{Q_2\}$	$Q_{2,3}$	$Q_{3,4}$	$Q_{4,5}$	$Q_{5,6}$	$\{Q_6\}$
Q_3	$\{Q_3\}$	$Q_{3,4}$	$Q_{4,5}$	$Q_{5,6}$	$\{Q_6\}$	
Q_4	$\{Q_4\}$	$Q_{4,5}$	$Q_{5,6}$	$\{Q_6\}$		
Q_5	$\{Q_5\}$	$Q_{5,6}$	$\{Q_6\}$			
Q_6	$\{Q_6\}$	$\{Q_6\}$				

Table 4
Operator D.

D	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
Q_1	$\{Q_1\}$					
Q_2	$\{Q_2\}$	$\{Q_2\}$				
Q_3	$\{Q_3\}$	$Q_{2,3}$	$\{Q_2\}$			
Q_4	$\{Q_4\}$	$Q_{3,4}$	$Q_{2,3}$	$\{Q_2\}$		
Q_5	$\{Q_5\}$	$Q_{4,5}$	$Q_{3,4}$	$Q_{2,3}$	$\{Q_2\}$	
Q_6	$\{Q_6\}$	$Q_{5,6}$	$Q_{4,5}$	$Q_{3,4}$	$Q_{2,3}$	$\{Q_2\}$

Appendix B. Proofs of propositions of sections 5–6

We suppose that A, B and C are subsets of Ω.

Proof of proposition 3. Let us suppose that $\Omega \subset_\beta B$ and $\Omega \subset_{\beta'} B \setminus A$. $A \subset B \Rightarrow B = A \cup B \setminus A$. A and $B \setminus A$ are disjoint. Using axiom 4, we obtain $Q_\beta \in \mathrm{S}(Q_\alpha, Q_{\beta'})$. Therefore $Q_\alpha \leqslant Q_\beta$. □

Proof of proposition 4. Let us suppose that $\Omega \subset_{\alpha'} A \setminus B$. $A = (A \cap B) \cup A \setminus B$ and $(A \cap B) \cap A \setminus B = \emptyset$. Using axiom 4, we obtain $Q_\alpha \in \mathrm{S}(Q_\lambda, Q_{\alpha'})$. Definition 7 gives us $Q_{\alpha'} \in \mathrm{D}(Q_\alpha, Q_\lambda)$. □

Proof of proposition 5. Let us suppose that $\Omega \subset_{\beta'} B \setminus A$. The proposition 4 gives us $Q_{\beta'} \in [\operatorname{Inf}\mathrm{D}(Q_\beta, Q_\lambda), \operatorname{Sup}\mathrm{D}(Q_\beta, Q_\lambda)]$. Since $A \cup B = A \cup (B \setminus A)$ and the sets A and $B \setminus A$ are disjoint, then according to axiom 4 if $\alpha + \beta' \leqslant M + 1$, $\Omega \subset_r A \cup B$ with $Q_r \in \mathrm{S}(Q_\alpha, Q_{\beta'})$. Definition 6 implies that $\operatorname{Inf}\mathrm{D}(Q_\beta, Q_\lambda) = Q_{\beta+1-\lambda} \leqslant Q_{\beta'} \leqslant \operatorname{Sup}\mathrm{D}\lambda(Q_\beta, Q_\lambda) = Q_{\beta+2-\lambda}$. Therefore, $\alpha + \beta' \leqslant M + 1 \Rightarrow \alpha + \beta + 1 - \lambda \leqslant \alpha + \beta' \leqslant \alpha + \beta + 2 - \lambda \leqslant M + 1$. So, we obtain (see table 5):

- If $\alpha + \beta + 2 - \lambda \leqslant M + 1 \Rightarrow \alpha + \beta + \lambda \leqslant M - 1$, then $Q_r \in [\operatorname{Inf}\mathrm{S}(Q_\alpha, \operatorname{Inf}\mathrm{D}(Q_\beta, Q_\lambda)), \operatorname{Sup}\mathrm{S}(Q_\alpha, \operatorname{Sup}\mathrm{D}(Q_\beta, Q_\lambda))]$.
- If $\alpha + \beta + 1 - \lambda = M + 1 \Rightarrow \alpha + \beta - \lambda \leqslant M$, then $Q_r \in \{\operatorname{Inf}\mathrm{S}(Q_\alpha, \operatorname{Inf}\mathrm{D}(Q_\beta, Q_\lambda))\} = \{Q_{M-1}\}$. □

Table 5
$\mathrm{U}(Q_\alpha, Q_\beta, Q_\lambda)$.

Q_α	Q_β	Q_λ	$Q_r \in \mathrm{U}(Q_\alpha, Q_\beta, Q_\lambda)$
Q_2	Q_2	Q_2	$[Q_2, Q_3]$
Q_3	Q_2	Q_2	$[Q_3, Q_4]$
Q_3	Q_3	Q_2	$[Q_3, Q_5]$
Q_3	Q_3	Q_3	$[Q_3, Q_4]$
Q_4	Q_2	Q_2	$[Q_4, Q_5]$
Q_4	Q_3	Q_2	$[Q_4, Q_6]$
Q_4	Q_3	Q_3	$[Q_4, Q_5]$
Q_4	Q_4	Q_2	$[Q_5, Q_6]$
Q_4	Q_4	Q_3	$[Q_4, Q_6]$
Q_4	Q_4	Q_4	$[Q_4, Q_5]$
Q_5	Q_2	Q_2	$[Q_5, Q_6]$
Q_5	Q_3	Q_2	$[Q_5, Q_6]$
Q_5	Q_3	Q_3	$[Q_5, Q_6]$
Q_5	Q_4	Q_2	$\{Q_6\}$
Q_5	Q_4	Q_3	$[Q_5, Q_6]$
Q_5	Q_4	Q_4	$[Q_5, Q_6]$
Q_5	Q_5	Q_3	$\{Q_6\}$
Q_5	Q_5	Q_4	$[Q_5, Q_6]$
Q_5	Q_5	Q_5	$[Q_5, Q_6]$
Q_6	Q_2	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_6	$\{Q_6\}$

Proof of corollary 6. Let us suppose $\Omega \subset_\lambda A \cap B$. Proposition 5 give us $Q_r \in \mathrm{U}(Q_\alpha, Q_\beta, Q_\lambda)$ if $\alpha + \beta - \lambda \leqslant M - 1$, and $Q_r \in \{Q_{M-1}\}$ if $\alpha + \beta - \lambda = M$. Thus we can deduce the values of Q_λ in accordance with those of Q_r, Q_α and Q_β, as this is showed in the table 6. From this table, we can verify that $Q_\lambda = Q_2$ if $\alpha + \beta - r = 1$ and $Q_\lambda \in [\mathrm{Inf}\,\mathrm{D}(Q_\beta, \mathrm{Sup}\,\mathrm{D}(Q_r, Q_\alpha)),\ \mathrm{Inf}\{\mathrm{Sup}\,\mathrm{D}(Q_\beta, \mathrm{Inf}\,\mathrm{D}(Q_r, Q_\alpha)), Q_\alpha, Q_\beta\}]$ otherwise. □

Proof of proposition 7 (Relative Duality). Let us suppose that $\Omega \subset_\alpha A$, $\Omega \subset_{\lambda 1} A \cap B$ and $A \subset_{\mu 2} A \setminus B$. We have $Q_{\mu 1} \in \mathrm{C}(Q_\alpha, Q_{\lambda 1})$ or equivalently $Q_{\lambda 1} = \mathrm{I}(Q_\alpha, Q_{\mu 1})$. Using proposition 4, we obtain $\Omega \subset_{\lambda 2} A \setminus B$ with $Q_{\lambda 2} \in \mathrm{D}(Q_\alpha, Q_{\lambda 1})$. For the different degrees of Q_α and $Q_{\mu 1}$, in $\mathfrak{L}_7$, the table 7 gives us the values of $Q_{\mu 2} \in \mathrm{C}(Q_\alpha, Q_{\lambda 2})$. We can verify that $Q_{n(\mu 1)+1} = Q_5 \in \mathrm{C}(Q_\alpha, Q_{\lambda 2})$, if $Q_\alpha = Q_4$ and $Q_{\mu 1} = Q_4$. For the other cases, we can verify that for any $Q_{\mu 1} \in \mathrm{C}(Q_\alpha, Q_{\lambda 1})$, there exists $\underline{Q_{\mu 2}} \in \mathrm{C}(\underline{Q_\alpha}, Q_{\lambda 2})$ such that $Q_{\mu 2} = Q_{n(\mu 1)}$. Since $A \setminus B = A \cap \overline{B}$, then $A \subset_{\mu 2} A \cap \overline{B}$ and, consequently, $A \subset_{\mu 2} \overline{B}$. □

Table 6

Q_r	Q_α	Q_β	$Q_\lambda \in$
Q_2	Q_2	Q_2	$\{Q_2\}$
Q_3	Q_2	Q_2	$\{Q_2\}$
Q_3	Q_3	Q_2	$\{Q_2\}$
Q_3	Q_3	Q_3	$[Q_2, Q_3]$
Q_4	Q_3	Q_2	$\{Q_2\}$
Q_4	Q_3	Q_3	$[Q_2, Q_3]$
Q_4	Q_4	Q_2	$\{Q_2\}$
Q_4	Q_4	Q_3	$[Q_2, Q_3]$
Q_4	Q_4	Q_4	$[Q_3, Q_4]$
Q_5	Q_3	Q_3	$\{Q_2\}$
Q_5	Q_4	Q_2	$\{Q_2\}$
Q_5	Q_4	Q_3	$[Q_2, Q_3]$
Q_5	Q_4	Q_4	$[Q_2, Q_4]$
Q_5	Q_5	Q_2	$\{Q_2\}$
Q_5	Q_5	Q_3	$[Q_2, Q_3]$
Q_5	Q_5	Q_4	$[Q_3, Q_4]$
Q_5	Q_5	Q_5	$[Q_4, Q_5]$
Q_6	Q_4	Q_3	$\{Q_2\}$
Q_6	Q_4	Q_4	$[Q_2, Q_3]$
Q_6	Q_5	Q_2	$\{Q_2\}$
Q_6	Q_5	Q_3	$[Q_2, Q_3]$
Q_6	Q_5	Q_4	$[Q_2, Q_4]$
Q_6	Q_5	Q_5	$[Q_3, Q_5]$
Q_6	Q_6	Q_3	$[Q_2, Q_3]$
Q_6	Q_6	Q_4	$[Q_3, Q_4]$
Q_6	Q_6	Q_5	$[Q_4, Q_5]$
Q_6	Q_6	Q_6	$[Q_5, Q_6]$

Proof of proposition 8 (Mixed Transitivity). Let us suppose $\Omega \subset_\alpha A$, $\Omega \subset_{\lambda 1} A \cap B$, $\Omega \subset_{\lambda 2} A \cap C$ and $A \subset_{\mu 2} C$. The definition 5 give us $Q_{\mu 1} \in \mathrm{C}(Q_\alpha, Q_{\lambda 1})$ and $Q_{\mu 2} \in \mathrm{C}(Q_\alpha, Q_{\lambda 2})$. $B \subset C \Rightarrow A \cap B \subset A \cap C$. Using proposition 3, we obtain $Q_{\lambda 1} \leqslant Q_{\lambda 2}$. We distinguish two cases:

(a) When $Q_{\lambda 1} < Q_{\lambda 2}$, table 1 of operator C implies that $Q_{\mu 1} < Q_{\mu 2}$.

(b) When $Q_{\lambda 1} = Q_{\lambda 2}$, we distinguish three cases:

(b1) if $A \cap B = A \cap C$, then $Q_{\mu 1} = Q_{\mu 2}$.

(b2) if $A \cap B \neq A \cap C = A$, then proposition 1 implies $Q_{\mu 1} < Q_M$ and $Q_{\mu 2} = Q_M$. Therefore, $Q_{\mu 1} < Q_{\mu 2}$.

(b3) if $A \cap B \neq A \cap C \neq A$, then:

- if $Q_{\lambda 1} \geqslant Q_3$ axiom 1 gives us $Q_{\mu 1} = Q_{\mu 2} = Q_{M-1}$ if $Q_\alpha = Q_{\lambda 1}$, and we have $\mathrm{Card}(\mathrm{C}(Q_\alpha, Q_{\lambda 1})) = 1$ otherwise (i.e., $Q_\alpha \neq Q_{\lambda 1} = Q_{\lambda 2}$). Therefore $Q_{\mu 1} = Q_{\lambda 2}$.

Table 7

Q_α	$Q_{\mu 1} \in$	a	b	c
Q_2	$[Q_2, Q_6]$	Q_2	$\{Q_2\}$	$[Q_2, Q_6]$
Q_3	$[Q_2, Q_5]$	Q_2	$[Q_2, Q_3]$	$[Q_2, Q_6]$
Q_3	$\{Q_6\}$	Q_3	$\{Q_2\}$	$[Q_2, Q_5]$
Q_4	$[Q_2, Q_4]$	Q_2	$[Q_3, Q_4]$	$[Q_5, Q_6]$
Q_4	$\{Q_5\}$	Q_3	$[Q_2, Q_3]$	$[Q_2, Q_5]$
Q_4	$\{Q_6\}$	Q_4	$\{Q_2\}$	$[Q_2, Q_4]$
Q_5	$[Q_2, Q_3]$	Q_2	$[Q_4, Q_5]$	$[Q_5, Q_6]$
Q_5	$\{Q_4\}$	Q_3	$[Q_3, Q_4]$	$[Q_4, Q_5]$
Q_5	$\{Q_5\}$	Q_4	$[Q_2, Q_3]$	$[Q_2, Q_4]$
Q_5	$\{Q_6\}$	Q_5	$\{Q_2\}$	$[Q_2, Q_3]$
Q_6	$\{Q_2\}$	Q_2	$[Q_5, Q_6]$	$[Q_5, Q_6]$
Q_6	$\{Q_3\}$	Q_3	$[Q_4, Q_5]$	$[Q_4, Q_5]$
Q_6	$\{Q_4\}$	Q_4	$[Q_3, Q_4]$	$[Q_3, Q_4]$
Q_6	$\{Q_5\}$	Q_5	$[Q_2, Q_3]$	$[Q_2, Q_3]$
Q_6	$\{Q_6\}$	Q_6	$\{Q_2\}$	$\{Q_2\}$

Note. (*a*) $Q_{\lambda 1} = \mathrm{I}(Q_\alpha, Q_{\mu 1})$; (*b*) $Q_{\lambda 2} \in \mathrm{D}(Q_\alpha, Q_{\lambda 1})$; (*c*) $Q_{\mu 2} \in \mathrm{C}(Q_\alpha, Q_{\lambda 2}) \setminus \{Q_7\}$.

- if $Q_{\lambda 1} = Q_2$ since we have $\mathrm{C}(Q_\alpha, Q_{\lambda 1}) = \mathrm{C}(Q_\alpha, Q_{\lambda 2})$, then for any $Q_{\mu 1} \in \mathrm{C}(Q_\alpha, Q_{\lambda 1})$ there exists $Q_{\mu 2} \in \mathrm{C}(Q_\alpha, Q_{\lambda 2})$ such that $Q_{\mu 1} \leqslant Q_{\mu - 2}$. □

Proof of proposition 9 (Exception). According to the classical contraposition $C \subset \overline{B} \Rightarrow B \subset \overline{C}$. $A \subset_\mu B$ and $B \subset \overline{C}$, then using proposition 7 we obtain $A \subset_\gamma \overline{C}$ with $Q_\mu \leqslant Q_\gamma$. Since we must not have $A \subset \overline{C}$, because this implies that $C \subset \overline{A}$ then this contradict the hypothesis $C \subset A$. Therefore, $Q_\gamma < Q_M$. □

Proof of proposition 10 (Union Right). Let us suppose that $A \subset_\mu B \cup C$. Since $A \subset_{\mu_1} B$, $B \subset B \cup C$, $A \subset_{\mu_2} C$ and $C \subset B \cup C$, then according to proposition 8 $Q_{\mu_1} \leqslant Q_\mu$ and $Q_{\mu_2} \leqslant Q_\mu$. Therefore $Q_{\mathrm{Max}(\mu_1, \mu_2)} \leqslant Q_\mu$. □

Proof of proposition 11 (Intersection Right). To show this proposition, we need to proove the following proposition:

Proposition 11bis. If $A \subset_{\mu_1} B$, $B \subset A$ and $A \subset C$, then $C \subset_{\mu_2} B$ with $Q_{\mu_2} \in [Q_2, Q_{\mu_1}]$. (This proposition is used to prove next propositions.)

Proof. Let us suppose that $\Omega \subset_{\alpha_1} A$, $\Omega \subset_\lambda B$, $\Omega \subset_{\alpha_2} C$ and $C \subset_{\mu_2} B$. $B \subset A \subset C \Rightarrow A \cap B = B \cap C = B$ and the definition 5 gives $Q_{\mu_1} \in \mathrm{C}(Q_{\alpha_1}, Q_\lambda)$ and $Q_{\mu_2} \in \mathrm{C}(Q_{\alpha_2}, Q_\lambda)$. Since $B \cap C = B \neq \emptyset$, proposition 2 implies that $Q_2 \leqslant Q_{\mu_2}$. $A \subset C$, then proposition 3 implies $Q_{\alpha_1} \leqslant Q_{\alpha_2}$. We distinguish two cases:

(a) $Q_{\alpha_1} < Q_{\alpha_2}$. Then:

- for $Q_\lambda \geqslant Q_3$, we can verify in the table 1 that for $Q_{\alpha_1} = Q_{M-1}$ and $Q_{\alpha_2} = Q_M$ we have $Q_{\mu_1} = Q_{\mu_2} = Q_\lambda$ and $Q_{\mu_2} < Q_{\mu_1}$ for the other cases.

- for $Q_\lambda = Q_2$, we can verify in the table 1 that for $\mathrm{Sup}\,\mathrm{C}(Q_{\alpha_2}, Q_2) \leqslant \mathrm{Sup}\,\mathrm{C}(Q_{\alpha_1}, Q_2)$ and $\mathrm{Inf}\,\mathrm{C}(Q_{\alpha_2}, Q_2) = \mathrm{Inf}\,\mathrm{C}(Q_{\alpha_1}, Q_2)$. Then for any $Q_{\mu_1} \in \mathrm{C}(Q_{\alpha_1}, Q_2)$, there is $Q_{\mu_2} \in \mathrm{C}(Q_{\alpha_2}, Q_2)$ such that $Q_{\mu_2} \leqslant Q_{\mu_1}$.

(b) $Q_{\alpha_1} = Q_{\alpha_2}$. We distinguish three cases:

(b1) $A = C$ implies that $Q_{\mu_1} = Q_{\mu_2}$.

(b2) $A = B \neq C$, this implies that $Q_{\mu_1} = Q_M$ and $Q_{\mu_2} < Q_M$. Therefore $Q_{\mu_2} < Q_{\mu_1}$.

(b3) $A \neq B$ and $A \neq C$, this implies that:

- for $Q_\lambda \geqslant Q_3$, if $Q_{\alpha_1} = Q_{\alpha_2} = Q_\lambda$, then [A3] gives $Q_{\mu_1} = Q_{\mu_2} = Q_{M-1}$ else (i.e., $Q_{\alpha_1} = Q_{\alpha_2} \neq Q_\lambda$) we have $\mathrm{Card}(\mathrm{C}(Q_{\alpha_1}, Q_\lambda)) = 1$ therefore $Q_{\mu_1} = Q_{\mu_2}$.
- for $Q_\lambda = Q_2$, since $\mathrm{C}(Q_{\alpha_1}, Q_2) = \mathrm{C}(Q_{\alpha_2}, Q_2)$, then for any $Q_{\mu_1} \in \mathrm{C}(Q_{\alpha_1}, Q_2)$ there is $Q_{\mu_2} \in \mathrm{C}(Q_{\alpha_2}, Q_2)$ such that $Q_{\mu_2} \leqslant Q_{\mu_1}$.

□

Back to proposition 11, let us suppose that $A \subset_\mu C$, $B \cap C \subset B$ and $B \cap C \subset C$. According to proposition 8 $A \subset_{\mu_1} B$ with $Q_\mu \leqslant Q_{\mu_1}$ and $A \subset_{\mu_2} C$ with $Q_\mu \leqslant Q_{\mu_2}$. Therefore $Q_\mu \leqslant Q_{\mathrm{Min}(\mu_1,\mu_2)}$. Let us suppose that $A \cap B \cap C = \emptyset$, then this implies that $A \cap B \subset A \setminus C$ and $A \cap C \subset A \setminus B$. We have $A \subset_{\mu_1} A \cap B$ and $A \subset_{\mu_2} A \cap C$. Proposition 11bis gives: $A \subset_{\gamma_1} A \setminus C$ and $A \subset_{\gamma_2} A \setminus B$ with $Q_{\mu_1} \leqslant Q_{\gamma_1}$ and $Q_{\mu_2} \leqslant Q_{\gamma_2}$. Proposition 7 gives $Q_{\gamma_2} = Q_{n(\mu_1)}$ if $Q_{\gamma_2} \neq Q_{n(\mu_1)}$ and $Q_{\gamma_2} \in [Q_{n(\mu_1)}, Q_{n(\mu_1)+1}]$ otherwise, and $Q_{\gamma_1} = Q_{n(\mu_2)}$ if $Q_{\gamma_1} \neq Q_{n(\mu_2)}$ and $Q_{\gamma_1} \in [Q_{n(\mu_2)}, Q_{n(\mu_2)+1}]$ otherwise. Therefore, if $A \cap B \cap C = \emptyset$, then $Q_{\mu_1} \leqslant Q_{n(\mu_2)}$ and $Q_{\mu_2} \leqslant Q_{n(\mu_1)}$. The contraposition of this consequence gives us: if $(Q_{\mu_1} > Q_{n(\mu_2)})$ or $(Q_{\mu_2} > Q_{n(\mu_1)})$, then $A \cap B \cap C \neq \emptyset$, this implies that $Q_\mu > Q_1$. □

Proof of proposition 12 (Mixed Union Left). Let us suppose that $A \cup B \subset_\gamma C$, $A \cup B \subset_{\gamma'} (A \cup B) \setminus C$ and $A \subset_{\mu'} A \setminus C$. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$. $B \subset C \Rightarrow (A \setminus C) \cup (B \setminus C) = A \setminus C$. Therefore, $A \cup B \subset_{\gamma'} A \setminus C$. Since $Q_\mu \leqslant Q_{M-1} \Rightarrow Q_{\gamma'} \neq Q_1 \Rightarrow Q_\gamma \leqslant Q_{M-1}$. $A \subset_{\mu'} A \setminus C$ and $A \cup B \subset_{\gamma'} A \setminus C$, then according to proposition 11bis: $\mu' \geqslant \gamma'$. Proposition 7 gives us: $(M + 1 \leqslant \mu + \mu' \leqslant M + 2)$ and $(M + 1 \leqslant \gamma + \gamma' \leqslant M + 2) \Rightarrow (\gamma - \mu) + (\gamma' - \mu') \geqslant 0$. Now $(\gamma' - \mu') \leqslant 0 \Rightarrow (\gamma - \mu) \geqslant 0$, that is $\mu \leqslant \gamma$. □

Proof of proposition 13 (Intersection/Product Syllogism). Let us suppose that $\Omega \subset_\alpha A$, $\Omega \subset_\lambda A \cap B$, $\Omega \subset_\delta A \cap B \cap C$ and $A \subset_\mu B \cap C$. We have $Q_\lambda = \mathrm{I}(Q_\alpha, Q_{\mu 1})$, $Q_\delta = \mathrm{I}(Q_\lambda, Q_{\mu 2})$ and $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\delta)$. From table 8, we can verify that for the different degrees of Q_α, Q_λ, Q_δ, $Q_{\mu 1}$ and $Q_{\mu 2}$ with $Q_\alpha \geqslant Q_\lambda \geqslant Q_\beta$ we have:

- if $Q_\delta \geqslant Q_3$, then $Q_\mu = \mathrm{I}(Q_{\mu 1}, Q_{\mu 2})$.
- if $Q_\delta = Q_2$, then $\mathrm{C}(Q_\alpha, Q_\delta) = [\mathrm{I}(\mathrm{Inf}\,\mathrm{C}(Q_\alpha, Q_\lambda), \mathrm{Inf}\,\mathrm{C}(Q_\lambda, Q_\delta)), \mathrm{I}(\mathrm{Sup}\,\mathrm{C}(Q_\alpha, Q_\lambda), \mathrm{Sup}\,\mathrm{C}(Q_\lambda, Q_\delta))]$.
  Since for any $Q_{\mu 1} \in \mathrm{C}(Q_\alpha, Q_\lambda)$ and $Q_{\mu 2} \in \mathrm{C}(Q_\lambda, Q_\delta)$, we have $\mathrm{I}(Q_{\mu 1}, Q_{\mu 2}) \in \mathrm{C}(Q_\alpha, Q_\delta)$. Therefore, $Q_\mu = \mathrm{I}(Q_{\mu 1}, Q_{\mu 2})$. □

Table 8

Q_α	Q_λ	Q_δ	*a*	*b*	*c*	*d*
Q_2	Q_2	Q_2	$[Q_2, Q_6]$	$[Q_2, Q_6]$	$[Q_2, Q_6]$	$[Q_2, Q_6]$
Q_3	Q_2	Q_2	$[Q_2, Q_5]$	$[Q_2, Q_6]$	$[Q_2, Q_5]$	$[Q_2, Q_5]$
Q_3	Q_3	Q_2	$\{Q_6\}$	$[Q_2, Q_5]$	$[Q_2, Q_5]$	$[Q_2, Q_5]$
Q_4	Q_3	Q_2	$\{Q_5\}$	$[Q_2, Q_5]$	$[Q_2, Q_4]$	$[Q_2, Q_4]$
Q_4	Q_4	Q_2	$\{Q_6\}$	$[Q_2, Q_4]$	$[Q_2, Q_4]$	$[Q_2, Q_4]$
Q_5	Q_3	Q_2	$\{Q_4\}$	$[Q_2, Q_5]$	$[Q_2, Q_3]$	$[Q_2, Q_3]$
Q_5	Q_4	Q_2	$\{Q_5\}$	$[Q_2, Q_4]$	$[Q_2, Q_3]$	$[Q_2, Q_3]$
Q_6	Q_4	Q_2	$\{Q_4\}$	$[Q_2, Q_4]$	$\{Q_2\}$	$\{Q_2\}$
Q_6	Q_6	Q_2	$\{Q_6\}$	$\{Q_2\}$	$\{Q_2\}$	$\{Q_2\}$
Q_3	Q_3	Q_3	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$
Q_4	Q_3	Q_3	$\{Q_5\}$	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$
Q_4	Q_4	Q_4	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$
Q_5	Q_4	Q_3	$\{Q_5\}$	$\{Q_5\}$	$\{Q_4\}$	$\{Q_4\}$
Q_5	Q_4	Q_4	$\{Q_5\}$	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$
Q_5	Q_5	Q_5	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$
Q_6	Q_5	Q_4	$\{Q_5\}$	$\{Q_5\}$	$\{Q_4\}$	$\{Q_4\}$
Q_6	Q_4	Q_5	$\{Q_5\}$	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$
Q_6	Q_5	Q_4	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$	$\{Q_5\}$
Q_6	Q_6	Q_5	$\{Q_6\}$	$\{Q6\}$	$\{Q_6\}$	$\{Q_6\}$

Note. (*a*) $Q_{\mu 1} \in \mathrm{C}(Q_\alpha, Q_\lambda) \setminus \{Q_7\}$; (*b*) $Q_{\mu 2} \in \mathrm{C}(Q_\lambda, Q_\delta) \setminus \{Q_7\}$; (*c*) $\mathrm{I}(Q_{\mu 1}, Q_{\mu 2})$; (*d*) $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\delta) \setminus \{Q_7\}$.

Proof of proposition 14 (Contraction). According to proposition 13 $A \subset_{\mu_1} B$ and $A \cap B \subset_{\mu_2} C \Rightarrow A \subset_\gamma B \cap C$ with $Q_\gamma = \mathrm{I}(Q_{\mu_1}, Q_{\mu_2})$. According to proposition 8 $A \subset_\gamma B \cap C$ and $B \cap C \subset C \Rightarrow A \subset_\mu C$ with $Q_\mu \geqslant Q_\gamma = \mathrm{I}(Q_{\mu_1}, Q_{\mu_2})$. If $Q_{\mu_2} \in [Q_2, Q_{M-1}]$, then $A \cap B \cap C \neq A \cap B$. This implies that $A \cap C \neq A$. Therefore, $Q_\mu \leqslant Q_{M-1}$. If $Q_{\mu_2} = Q_M$, then $Q_\mu \leqslant Q_M = Q_{\mu_2}$. □

Proof of proposition 15 (Intersection/Quotient syllogism). Proposition 13 gives $A \subset_{\mu'} B \cap C$ with $Q_{\mu'} = \mathrm{I}(Q_{\mu_1}, Q_{\mu_3})$. Let us suppose that $\Omega \subset_\alpha A, \Omega \subset_\beta C$ and $\Omega \subset_\lambda A \cap B \cap C$. Then, we have $Q_\beta = \mathrm{I}(Q_\alpha, Q_{\mu_2})$ and $Q_\lambda = \mathrm{I}(Q_\alpha, Q_{\mu'})$. Let us suppose that $A \cap C \subset_\mu B$, then $Q_\mu \in \mathrm{C}(Q_\beta, Q_\lambda) = \mathrm{C}(\mathrm{I}(Q_\alpha, Q_{\mu_2}), \mathrm{I}(Q_\alpha, Q_{\mu'}))$. We obtain (table 9):

- For any $Q_\lambda = \mathrm{I}(Q_\alpha, Q_{\mu'}) \geqslant Q_3, \mathrm{C}(\mathrm{I}(Q_\alpha, Q_{\mu_2}), \mathrm{I}(Q_\alpha, Q_{\mu'})) = \mathrm{C}(Q_{\mu_2}, Q_{\mu'})$;
- For any $Q_\lambda = Q_2, \mathrm{C}(Q_{\mu_2}, Q_{\mu'}) \subset \mathrm{C}(\mathrm{I}(Q_\alpha, Q_{\mu_2}), \mathrm{I}(Q_\alpha, Q_{\mu'}))$ and therefore $\mathrm{C}(\mathrm{I}(Q_\alpha, Q_{\mu_2}), \mathrm{I}(Q_\alpha, Q_{\mu'}))$ can be restricted to $\mathrm{C}(Q_{\mu_2}, Q_{\mu'})$ for verifying the property $\mathrm{C}(\mathrm{I}(Q_\alpha, Q_{\mu_2}), \mathrm{I}(Q_\alpha, Q_{\mu'})) = \mathrm{C}(Q_{\mu_2}, Q_{\mu'})$. Therefore, $Q_\mu \in \mathrm{C}(Q_{\mu_2}, Q_{\mu'}) = \mathrm{C}(Q_{\mu_2}, \mathrm{I}(Q_{\mu_1}, Q_{\mu_3}))$. $Q_{\mu_1} < Q_M \Rightarrow A \neq A \cap B \Rightarrow A \cap C \neq A \cap B \cap C$. So, $Q_\mu < Q_M$. Therefore, $Q_\mu \in \mathrm{C}(Q_{\mu_2}, \mathrm{I}(Q_{\mu_1}, Q_{\mu_3})) \setminus \{Q_M\}$. □

Proof of proposition 16 (Weak Transitivity). $B \subset_{\mu_2} C$ and $B \subset A \Rightarrow A \cap B \subset_{\mu_2} C$. Proposition 13 gives $A \subset_{\mu'} B \cap C$ with $Q_{\mu'} = \mathrm{I}(Q_{\mu_1}, Q_{\mu_2})$. Since $B \cap C \subset C$, then according to proposition 11bis $A \subset_{\mu'} C$ with $Q_{\mu'} = \mathrm{I}(Q_{\mu_1}, Q_{\mu_2}) \leqslant Q_\mu$. If

Table 9

Q_α	Q_{μ_2}	$Q_{\mu'}$	$Q_\beta =$ $\mathrm{I}(Q_\alpha, Q_{\mu_2})$	$Q_\lambda =$ $\mathrm{I}(Q_\alpha, Q_{\mu'})$	$\mathrm{C}(\mathrm{I}(Q_\alpha, Q_{\mu_2}),$ $\mathrm{I}(Q_\alpha, Q_{\mu'})) \setminus \{Q_7\}$	$\mathrm{C}(Q_{\mu_2}, Q_{\mu'}) \setminus \{Q_7\}$
Q_α	Q_{μ_2}	Q_{μ_2}	Q_β	Q_β	$[Q_a, Q_b]$	$[Q_a, Q_b]$
Q_3	Q_3	Q_2	Q_2	Q_2	$[Q_2, Q_6]$	$[Q_2, Q_5]$
Q_4	Q_3	Q_2	Q_2	Q_2	$[Q_2, Q_6]$	$[Q_2, Q_5]$
Q_4	Q_4	Q_3	Q_2	Q_2	$[Q_2, Q_6]$	$\{Q_5\}$
Q_4	Q_5	Q_3	Q_3	Q_2	$[Q_2, Q_5]$	$\{Q_4\}$
Q_4	Q_5	Q_4	Q_3	Q_2	$[Q_2, Q_5]$	$\{Q_5\}$
Q_4	Q_6	Q_5	Q_4	Q_3	$\{Q_5\}$	$\{Q_5\}$
Q_5	Q_5	Q_3	Q_4	Q_2	$[Q_2, Q_4]$	$\{Q_4\}$
Q_5	Q_5	Q_4	Q_4	Q_3	$\{Q_4\}$	$\{Q_4\}$
Q_5	Q_6	Q_2	Q_5	Q_2	$[Q_2, Q_3]$	$\{Q_2\}$
Q_5	Q_6	Q_3	Q_5	Q_2	$[Q_2, Q_3]$	$\{Q_3\}$
Q_5	Q_6	Q_4	Q_5	Q_3	$\{Q_5\}$	$\{Q_5\}$
Q_5	Q_6	Q_5	Q_5	Q_4	$\{Q_5\}$	$\{Q_5\}$
Q_6	Q_6	$Q_{\mu'}$	Q_6	$Q_{\mu'}$	$\{Q_{\mu'}\}$	$\{Q_{\mu'}\}$

$Q_{\mu_2} \in [Q_2, Q_{M-1}]$, then $B \cap C \neq C \Rightarrow A \cap C \neq C$, since $B \subset A$. Therefore $Q_\mu \leqslant Q_{M-1}$. If $Q_{\mu_2} = Q_M$, then $Q_\mu \leqslant Q_M = Q_{\mu_2}$. □

Proof of corollary 17 (Contraction). It is a consequence of proposition 14. □

Proof of proposition 18 (Cumulativity). Let us suppose that $\Omega \subset_\alpha A$, $\Omega \subset_\lambda A \cap B \cap C$ and $A \cap B \subset_\mu C$. We have $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda)$. Axiom 2 implies that $(A \subset_{M-1} B \Rightarrow \Omega \subset_\alpha A \cap B)$ and $(A \subset_{M-1} C \Rightarrow \Omega \subset_\alpha A \cap C)$ and proposition 1 implies that $A \cap B \neq A$ and $A \cap C \neq A$. Since $A \cap B \subset (A \cap B) \cup (A \cap C) \subset A$, $\Omega \subset_\alpha A$ and $\Omega \subset_\alpha A \cap B$, then $\Omega \subset_\alpha (A \cap B) \cup (A \cap C)$. Corollary 6 give us $Q_\lambda \in [\mathrm{Inf}\,\mathrm{D}(Q_\alpha, \mathrm{Sup}\,\mathrm{D}(Q_\alpha, Q_\alpha)), \mathrm{Sup}\,\mathrm{D}(Q_\alpha, \mathrm{Inf}\,\mathrm{D}(Q_\alpha, Q_\alpha))]$ which is equal to $[Q_{\alpha-1}, Q_\alpha]$ for $\alpha \geqslant 3$ and to $\{Q_2\}$ for $\alpha = 2$. Therefore, $Q_\mu \in \mathrm{C}(Q_\alpha, Q_\lambda)$ which is equal to $([\mathrm{Inf}\,\mathrm{C}(Q_\alpha, Q_{\alpha-1}), \mathrm{Sup}\,\mathrm{C}(Q_\alpha, Q_\alpha)])$ for $\alpha \geqslant 3$ and to $\mathrm{C}(Q_2, Q_2)$ for $\alpha = 2$. It is evident to verify in table 1 of C that:

- for $\alpha \geqslant 4$, $[\mathrm{Inf}\,\mathrm{C}(Q_\alpha, Q_{\alpha-1}), \mathrm{Sup}\,\mathrm{C}(Q_\alpha, Q_\alpha] = [Q_{M-2}, Q_M]$;
- for $2 \leqslant \alpha \leqslant 3$, $\mathrm{C}(Q_\alpha, Q_\lambda) = [Q_2, Q_M]$. Since for $\alpha \geqslant 4$, we have $Q_\mu \in [Q_{M-2}, Q_M]$, then for $2 \leqslant \alpha \leqslant 3$ the interval $[Q_2, Q_M]$ can be restrained to the interval $[Q_{M-2}, Q_M]$. Therefore $Q\mu \in [Q_{M-2}, Q_M]$. □

Proof of proposition 19 (Union Left). Let us suppose that $\Omega \subset_\alpha A, \Omega \subset_\beta B, \Omega \subset_{\lambda 1} A \cap B$ and $\Omega \subset_{\lambda 2} A \cap B \cap C$. According to axiom 2 $(A \subset_{M-1} C$ and $\Omega \subset_\alpha A \Rightarrow \Omega \subset_\alpha A \cap C)$ and $(B \subset_{M-1} C$ and $\Omega \subset_\beta B \Rightarrow \Omega \subset_\beta B \cap C)$. Proposition 1 implies that $A \cap C \neq A$ and $B \cap C \neq B$. We have $(A \cup B) \cap C = (A \cap C) \cup (B \cap C))$. Then, proposition 5 gives us:

- $\Omega \subset_{r1} A \cup B$ with $Q_{r1} \in U(Q, Q, Q) = [\mathrm{Inf}\,\mathrm{S}(Q_\alpha, \mathrm{Inf}\,\mathrm{D}(Q_\beta, Q_{\lambda 1})), \mathrm{Sup}\,\mathrm{S}(Q_\alpha, \mathrm{Sup}\,\mathrm{D}(Q_\beta, Q_{\lambda 1}))]$ if $\alpha + \beta - \lambda 1 \leqslant M - 1$, and $Q_{r1} = Q_{M-1}$ if $\alpha + \beta - \lambda 1 = M$;

Table 10

Q_r	Q_α	Q_β	$Q_\lambda \in$
Q_2	Q_2	Q_2	$[Q_2, Q_3]$
Q_3	Q_2	Q_2	$[Q_3, Q_4]$
Q_3	Q_3	Q_2	$[Q_3, Q_5] \setminus \{Q_5\}$
Q_3	Q_3	Q_3	$[Q_3, Q_4]$
Q_4	Q_2	Q_2	$[Q_4, Q_5]$
Q_4	Q_3	Q_2	$[Q_4, Q_6] \setminus \{Q_6\}$
Q_4	Q_3	Q_3	$[Q_4, Q_5]$
Q_4	Q_4	Q_2	$[Q_5, Q_6] \setminus \{Q_6\}$
Q_4	Q_4	Q_3	$[Q_4, Q_6] \setminus \{Q_4, Q_6\}$
Q_4	Q_4	Q_4	$[Q_4, Q_5] \setminus \{Q_4\}$
Q_5	Q_2	Q_2	$[Q_5, Q_6]$
Q_5	Q_3	Q_2	$[Q_5, Q_6]$
Q_5	Q_3	Q_3	$[Q_5, Q_6]$
Q_5	Q_4	Q_3	$[Q_5, Q_6] \setminus \{Q_5\}$
Q_5	Q_4	Q_4	$[Q_5, Q_6] \setminus \{Q_5\}$
Q_5	Q_5	Q_3	$\{Q_6\}$
Q_5	Q_5	Q_4	$[Q_5, Q_6] \setminus \{Q_5\}$
Q_5	Q_5	Q_5	$[Q_5, Q_6] \setminus \{Q_5\}$
Q_6	Q_2	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_6	$\{Q_6\}$

- $\Omega \subset_{r2} (A \cap C) \cup (B \cap C)$ with $Q_{r2} \in \mathrm{U}(Q_\alpha, Q_\beta, Q_{\lambda 2}) = [\mathrm{Inf\,S}(Q_\alpha, \mathrm{Inf\,D}(Q_\beta, Q_{\lambda 2})), \mathrm{Sup\,S}(Q_\alpha, \mathrm{Sup\,D}(Q_\beta, Q_{\lambda 2}))]$ if $\alpha + \beta - \lambda 2 \leqslant M - 1$, and $Q_{r2} = Q_{M-1}$ if $\alpha + \beta - \lambda 2 = M$.

Since $(A \cup B) \cap C \subset A \cup B$ and $A \cap B \cap C \subset A \cap B$, then proposition 3 gives us $Q_{r2} \leqslant Q_{r1}$ and $Q_{\lambda 2} \leqslant Q_{\lambda 1}$.

The following constraint: for any $Q_{\lambda 2}, Q_{\lambda 1}$ such that $Q_{\lambda 2} \leqslant Q_{\lambda 1}$ we must have $Q_{r2} \leqslant Q_{r1}$, allow us to suppress each value of $\mathrm{U}(Q_\alpha, Q_\beta, Q_{\lambda 1})$ and $\mathrm{U}(Q_\alpha, Q_\beta, Q_{\lambda 2})$ for which $Q_{r2} > Q_{r1}$ as that is showed in table 10. From this table, we can verify that for any $Q_\alpha, Q_\beta, Q_{\lambda 2}, Q_\lambda$ with $Q_{\lambda 2} \leqslant Q_{\lambda 1}$, we have $Q_{r2} = Q_{r1}$ or $Q_{r2} = Q_{r1-1}$.

If $A \cup B \subset_\mu C$, then $Q_\mu \in \mathrm{C}(Q_{r1}, Q_{r2})$. We have $(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \neq A \cup B$ (as $A \cap C \neq A$ and $B \cap C \neq B$), then $Q_\mu \in \mathrm{C}(Q_{r1}, Q_{r2}) \setminus \{Q_M\}$ which is equal to $([\mathrm{Inf\,C}(Q_{r1}, Q_{r1-1}), \mathrm{Sup\,C}(Q_{r1}, Q_{r1})]) \setminus \{Q_M\}$ for $Q_{r1} \geqslant Q_3$, and to $\mathrm{C}(Q_2, Q_2) \setminus \{Q_M\}$ for $Q_{r1} = Q_2$. It is obvious to verify in table 1 of C that:

- for $Q_{r1} \geqslant Q_4$, $[\mathrm{Inf\,C}(Q_{r1}, Q_{r1-1}),\ \mathrm{Sup\,C}(Q_{r1}, Q_{r1})] \setminus \{Q_M\} = [Q_{M-2}, Q_{M-1}]$;
- for $Q_2 \leqslant Q_{r1} \leqslant Q_3$, $\mathrm{C}(Q_{r1}, Q_{r2}) \setminus \{Q_M\} = [Q_2, Q_{M-1}]$. Since for $Q_{r1} \geqslant Q_4$,

$Q_\mu \in [Q_{M-2}, Q_{M-1}]$, then for $Q_2 \leqslant Q_{r1} \leqslant Q_3$ the interval $[Q_2, Q_{M-1}]$ can be restrained to the interval $[Q_{M-2}, Q_{M-1}]$.
Therefore, $Q_\mu \in [Q_{M-2}, Q_{M-1}]$. □

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